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Title: On bipartite $(1, 1, k)$-mixed graphs.

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Review text:

The degree/diameter problem asks to find the maximum order of a graph with maximum degree Δ and diameter at most D. A straightforward calculation gives an upper bound, called the Moore bound, for the order of such a graph, and graphs that attain this bound are called Moore graphs. A longstanding open problem is to settle the question of existence of a Moore graph of diameter $D = 2$ and maximum degree $\Delta = 57$. The degree/diameter problem has also been studied for directed graphs, and more recently for mixed graphs, that is, graphs which are allowed to have undirected edges as well as directed arcs; see Miller–Širáň [Electron. J. Combin. **DS14** (2005), 61 pp.; MR4336216] for a dynamic survey of the area.

The current work focusses on bipartite (r, z)-mixed graphs, that is, mixed graphs with maximum undirected degree r and maximum directed out-degree z, in the case when $r = 1 = z$. It follows from the work of Dalfó–Fiol–López [Discrete Appl. Math. **219** (2017), 110–116; MR3599462] that the Moore bound for such graphs with diameter k grows as $\Theta(\varphi^k)$, where φ is the golden ratio $(1 + \sqrt{5})/2$. The same authors also found extremal graphs for $k = 3$ and $k = 4$. In this article, an extremal example for $k = 5$ is found by the third author through an exhaustive computer search, and further examples (which are not known to be extremal) for $k = 6, \dots, 16$ are constructed using Cayley graphs and their voltage lifts; a GAP search helped find viable candidates for such Cayley graphs.

Next, the authors construct an infinite family of bipartite $(1, 1)$-mixed graphs with diameter k and order $\Theta(2^{k/2})$. They also show that the construction can be modified so that the graphs are totally regular, that is, every vertex has undirected degree as well as directed out-degree equal to 1. These are generalizations of a construction of Fiol–Yebra [J. Graph Theory **14** (1990), no. 6, 687–700; MR1079217] for bipartite digraphs.

Next, the authors construct a family of bipartite $(1,1)$-mixed graphs that arise from (duals of) regular triangulations of the torus, and these are called chordal ring mixed graphs in this article. A Moore bound is computed for such graphs, and extremal examples are found and described using plane tessellations.

Lastly, the authors find an improved Moore bound for bipartite $(1,1)$-mixed graphs using a Moore tree construction of Buset–El Amiri–Erskine–Miller–Pérez-Rosés [Discrete Math. **339** (2016), no. 8, 2066–2069; MR3500136].

The exposition in the article is clear and easy to follow. The figures are carefully designed so that they can be interpreted even without color. Notations are sometimes reused, but the context usually clarifies the meaning. Full details are sometimes omitted: in the proof of Proposition 3.2, OEIS data is used to infer a general expression for $v(n)$ and $u(n)$, and in the proof of Theorem 4.2, the optimal tiles (as in Figure 9 or Figure 11) are visually clear, but full proofs are avoided.

There are also a few minor typos. The bottom-left vertex in the graph on the left in Figure 3 is missing its label, $(0,3)_1$. In section 3, the graph $BDM(2,m)$ for $m = 2^{n-1} + 2^{n-3}$ should be defined only for $n \geq 3$ instead of $n \geq 2$. Near the bottom of page 7, the term $(0, i+m/2)$ is missing its subscript, so it should read $(0, i+m/2)_0$. In the proof of (ii) in Lemma 3.1, in the first displayed equation, the term $(1, 2i+1+2^{n+2})_0$ should be $(1, 2i+1+2^{n-2})_0$. In Figure 9 and its description at the bottom of page 16, the labels on the triangles should run from 0 to 7, instead of 1 to 8.